Gravitational clustering in a D-dimensional Universe.

T. Padmanabhan[1*] & Nissim Kanekar[2†]
1 *Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India*
2 *National Centre for Radio Astrophysics, Post Bag 3, Ganeshkhind, Pune 411 007, India*

We consider the problem of gravitational clustering in a D-dimensional expanding Universe and derive scaling relations connecting the exact mean two-point correlation function with the linear mean correlation function, in the quasi-linear and non-linear regimes, using the standard paradigms of scale-invariant radial collapse and stable clustering. We show that the existence of scaling laws is a generic feature of gravitational clustering in an expanding background, in all dimensions except $D = 2$ and comment on the special nature of the 2-dimensional case. The D-dimensional scaling laws derived here reduce, in the 3-dimensional case, to scaling relations obtained earlier from N-body simulations. Finally, we consider the case of clustering of 2-dimensional particles in a 2-D expanding background, governed by a force $-GM/R$, and show that the correlation function does not grow (to first order) until much after the recollapse of any shell.

I. Introduction

The temporal evolution of a system consisting of a large number of particles interacting with each other via Newtonian gravity is a formidable problem to tackle. Such a system has, in fact, no "final" state of thermodynamic equilibrium as there is no limit to its phase space volume. The phase volume available can be continuously increased by the separation of particles into a collapsed core and a dispersed halo, with the core becoming more and more tightly bound and the halo particles moving to larger and larger distances. In fact, the only stable configuration of such a system consists of a tightly coupled binary, with all the other particles at infinite distance.
The above situation changes drastically if the background of the system (*i.e.* the space in which the particles are moving) is itself expanding. In this case, the expansion provides a "stabilising" influence on the evolution and can result in the formation of stable structures in which the effects of gravitational collapse are balanced by the expansion. Further, there also exists the possibility that the outmoving halos may be captured by other compact cores, leading to the build up of larger structures. Thus, the issue of gravitational clustering of a large number of particles appears more tractable in the case of a background expanding (in general) with a time-dependant scale factor. This problem is, of course, of immense physical relevance as there currently exists strong evidence that the matter density in the Universe is dominated by collisionless dark matter particles, which interact solely by gravity. Thus, if the length scales of interest are much smaller than the Hubble scale (and particle velocities are non-relativistic), the formation of large-scale structure in the Universe is well described by the above picture, making it worthy of investigation.
In the linear regime, where deviations from uniformity are small, perturbative techniques are used to obtain the time evolution of the system parameters, for example, the correlation functions. These methods, of course, fail in the quasi-linear and non-linear regimes and there is, as yet, no clear analytical picture of the behaviour of the system in these stages. However, in the case of 3+1 dimensions, there exists a set of scaling relations relating the linear and non-linear correlation functions ([2], [4]). These scaling laws are *not* particularly well understood but appear to be validated by numerical simulations ([2], [5]).
A better understanding of a physical problem is sometimes attained by treating it in a more general manner in an arbitrary number of dimensions as one may then be able to separate the generic features of the problem (as in, issues arising from the nature of the interaction) from results which stem from its dimensionality. This has been seen earlier, for example, in the case of the Ising model. In the current letter, we address the issue of gravitational clustering in a D-dimensional, expanding Universe and attempt to derive the scaling relations, using the well-known paradigms of scale-invariant radial infall in the quasi-linear phase and stable clustering in the non-linear regime.

II. Clustering in D-dimensions

We consider the case of a D-dimensional Universe, expanding with a time-dependant scale factor $a(t)$. Further, we use the maximally symmetric Robertson-Walker metric (in $D+1$ dimensions) and specialise to flat space ($k = 0$). The equations governing the evolution of $a(t)$ are then (see [8])

$$\frac{\dot{a}^2}{a^2} = \frac{2\kappa(D)}{D(D-1)}\rho_b \tag{1}$$

$$\frac{\ddot{a}}{a} + \frac{D-2}{2}\left(\frac{\dot{a}}{a}\right)^2 = -\frac{\kappa(D)}{(D-1)}p \tag{2}$$

where ρ_b and p are, respectively, the density and pressure of the background Universe and κ is the constant in the Einstein equations, which, in general, can depend on the dimension D. We note that equations (1) and (2) are obtained by imposing the constraints of homogeneity and isotropy on the Einstein equations in $(D+1)$ dimensions;

the situation is thus manifestly isotropic. Further, it can be seen that the behaviour in the $D = 2$ case is likely to be special, due to the presence of the $(D-2)$ coefficient in the second term of equation (2). We emphasise that the present work considers the clustering of D-dimensional particles in a D-dimensional expanding Universe. Earlier studies of 2-D clustering in the literature (see, for example, [7]) treated the clustering of infinite needles in a 3-dimensional background; this will be commented upon later.

Given an equation of state, $p(\rho_b)$, one can solve equations (1) and (2) for the evolution of $a(t)$. For pressureless dust, with $p = 0$, this implies that ρ_b is given by

$$\rho_b \propto a^{-D} \tag{3}$$

Next, we incorporate the effects of perturbations on the smooth background by considering the evolution of the above system starting from Gaussian initial conditions with an initial power spectrum $P_{in}(k)$. The two-point correlation function, $\xi(a, x)$, is defined as the Fourier transform of the power spectrum. We will, for convenience, work with the *mean* two-point correlation function, $\bar{\xi}(a, x)$, defined by

$$\bar{\xi}(a, x) = \frac{D}{x^D} \int_0^x dy y^{D-1} \xi(a, y) \tag{4}$$

and attempt to relate the exact $\bar{\xi}(a, x)$ in the quasi-linear and non-linear regimes to the mean correlation function calculated from linear theory, at the same epoch.

The equation for the conservation of pairs can be written in D-dimensions as

$$\frac{\partial \xi}{\partial t} + \frac{1}{a x^{D-1}} \frac{\partial}{\partial x} \left[x^{D-1} v (1 + \xi) \right] = 0 \tag{5}$$

where $v(a, x)$ is the mean relative pair velocity at scale x and time t. In terms of $\bar{\xi}(a, x)$, this gives

$$\left[\frac{\partial}{\partial A} - h(a, x) \frac{\partial}{\partial X} \right] \ln(1 + \bar{\xi}) = D h(a, x) \tag{6}$$

Here, $h(a, x) = -v/\dot{a}x$, $X = \ln x$ and $A = \ln a$. The above equation can be further simplified by defining $F = \ln \left[x^D (1 + \bar{\xi}) \right]$, yielding

$$\left[\frac{\partial}{\partial A} - h(a, x) \frac{\partial}{\partial X} \right] F = 0 \tag{7}$$

The characteristic curves of this equation, on which F is a constant, satisfy the condition $\ln \left[x^D (1 + \bar{\xi}) \right] =$ constant, *i.e.*

$$x^D (1 + \bar{\xi}) = l^D \tag{8}$$

where l is some other length scale. (Note that $x^D(1 + \bar{\xi})$ is proportional to the number of neighbours of a given particle, within a D-dimensional sphere of radius ax; the above equation expresses the conservation of pairs in this sphere [9].) When the evolution is linear at all relevant scales, $\bar{\xi}(a, x) \ll 1$ and $x \approx l$. However, in the non-linear regime, $\bar{\xi}(a, x) \gg 1$ at some scale x; clearly $x \ll l$. Thus, the behaviour of clustering at a scale x is determined by the transfer of power from a larger scale l along the characteristics defined by equation (7). This suggests that one should try to relate the true $\bar{\xi}(a, x)$ to the linear correlation function $\bar{\xi}_L(a, l)$, evaluated at a *different* scale.

The paradigm of scale-invariant radial collapse (see [1], [3]) will be used to carry out the above procedure, in the quasi-linear regime. Consider the evolution of a D-dimensional, spherically symmetric, overdense region, containing a mass M. In general, such a region will initially expand with the background Universe until the excess gravitational force due to its enclosed mass causes it to collapse back upon itself. Thus, the radius of the region will initially rise, reach a maximum and then decrease. The equation of motion for the radius, $R(t)$, of such a D-dimensional, spherical region is ([8])

$$\frac{d^2 R}{dt^2} = -\frac{2(D-2)}{D^2} (1 + \delta_i) \frac{l^2}{t_i^2} \frac{l^{D-2}}{R^{D-1}} \tag{9}$$

where we have replaced for the mass, M, in terms of δ_i, l and t_i, *i.e.* the initial density contrast, shell radius and time, respectively. We note that the acceleration, $(d^2 R/dt^2)$, is proportional to $1/R^{D-1}$, in D dimensions. The energy integral for equation (9) is

$$E = \frac{1}{2} \left(\frac{dR}{dt} \right)^2 - \frac{2(1 + \delta_i)}{D^2} \frac{l^2}{t_i^2} \left(\frac{l}{R} \right)^{D-2} \tag{10}$$

The above expression is, of course, not valid for $D = 2$; this case will be treated separately later. We evaluate the energy constant, E, by requiring that the velocity satisfies the unperturbed expansion at $t = t_i$, *i.e.* $dR/dT = l H_i = 2l/Dt_i$, where $H_i = 2/Dt_i$ is the Hubble parameter. This gives

$$\left(\frac{dR}{dt} \right)^2 = \frac{4(1 + \delta_i)}{D^2} \frac{l^2}{t_i^2} \left(\frac{l}{R} \right)^{D-2} - \frac{4\delta_i}{D^2} \frac{l^2}{t_i^2} \tag{11}$$

At turnaround, $R = R_M$ and $dR/dt = 0$. Thus

$$\left(\frac{\delta_i}{1 + \delta_i} \right) = \left(\frac{l}{R_M} \right)^{D-2} \tag{12}$$

or, since $\delta_i \ll 1$,

$$R_M = l\delta_i^{1/(2-D)} \tag{13}$$

In the quasi-linear regime, we expect clustering to take place in a region surrounding density peaks of the linear regime, *i.e.* around regions such as the spherical region considered above. Making the usual assumption that the typical density profile around such a peak is equal to the average profile around a mass point, we can write this density profile as

$$\rho(x) \approx \rho_{\rm b}(1+\bar{\xi}) \tag{14}$$

Thus, the initial density contrast, $\delta_i(l) \propto \bar{\xi}_L(l)$, in the initial epoch, when linear theory is valid. Since $R_M = l\delta_i^{1/(2-D)}$, clearly $R_M \propto \bar{\xi}_L^{1/(2-D)}$. In the scale-invariant, radial collapse picture, each shell can be approximated as contributing an effective radius proportional to R_M. Taking the final effective radius, x, as proportional to R_M, the final mean correlation function is given by

$$\bar{\xi}(x) \propto \rho \propto \frac{M}{x^D} \propto \frac{l^D}{\left[l\bar{\xi}_L^{1/(2-D)}\right]^D} \propto \left[\bar{\xi}_L(l)\right]^{D/(D-2)} \tag{15}$$

Thus, the final correlation function $\bar{\xi}_{\rm QL}$ at x is the $D/(D-2)$ power of the initial correlation function at l, where $l^D \propto x^D\left[\bar{\xi}_L(l)\right]^{D/(D-2)} \propto x^D\bar{\xi}_{\rm QL}(x)$, which is the form required by equation (8), if $\bar{\xi}_{\rm QL} \gg 1$.

Let us turn our attention, next, to the non-linear regime; here, we use the ansatz of stable clustering ($h \to 1$ for $\bar{\xi} \gg 1$), which is physically well-motivated as it seems reasonable to expect stable, bound systems to form under the joint influence of gravity and the expansion ([9]). Such systems would neither expand nor contract and would hence have peculiar velocities equal and opposite to the Hubble expansion, *i.e.* $v^i = -\dot{a}x^i$. (We emphasise that the stable clustering hypothesis is an *ansatz* and might well fail if mergers of structures are important.) Stable clustering requires that virialized systems should maintain their densities and sizes in *proper* co-ordinates, $r = ax$. This would require the correlation function, in D dimensions, to have the form $\bar{\xi}_{\rm NL}(a,x) = a^D F(ax)$, where F is some function and the a^D factor arises from the decrease in the background density. We assume that $\bar{\xi}_{\rm NL}(a,x)$ is a function of $\bar{\xi}_L(a,l)$, where $l^D \approx x^D\bar{\xi}_{\rm NL}(a,x)$, from equation (8). This relation can be written as

$$\bar{\xi}_{\rm NL}(a,x) = a^D F(ax) = U\left[\bar{\xi}_L(a,l)\right] \tag{16}$$

where $U(p)$ is an unknown function of its argument, which is to be determined. The density contrast and mean correlation function have the relation $\bar{\xi} \propto \delta^2$, in the linear regime; this arises from the definition of ξ as the Fourier transform of δ^2. Now, in a flat D-dimensional matter-dominated Universe, $\delta \propto a^{D-2}$ ([8]), in the linear regime. Thus, $\bar{\xi}_L \propto \delta_l^2 \propto a^{2(D-2)}$. We can hence write $\bar{\xi}_L(a,l) = a^{2(D-2)}Q\left[l^D\right]$, where Q is a function of l alone. But, $l^D \approx x^D\bar{\xi}_{\rm NL}(a,x) = x^D a^D F(ax) = r^D F(r)$. This implies

$$\bar{\xi}_{\rm NL}(a,x) = a^D F(r) = U\left[\bar{\xi}_L(a,l)\right] \tag{17}$$

$$= U\left[a^{2(D-2)}Q\left[r^D F(r)\right]\right] \tag{18}$$

Considering the above relation as a function of a at constant r, we obtain

$$Ba^D = F\left[Ca^{2(D-2)}\right] \tag{19}$$

where B and C are constants. One must hence have

$$F(p) \propto p^{D/2(D-2)} \tag{20}$$

Thus, in the extreme non-linear regime, we obtain

$$\bar{\xi}_{\rm NL}(a,x) \propto \left[\bar{\xi}_L(a,l)\right]^{D/2(D-2)} \tag{21}$$

The above analysis shows that the exact mean correlation function can be expressed in terms of the linear mean correlation function by the relation

$$\bar{\xi}(a,x) = \bar{\xi}_L(a,l) \qquad \text{(linear)} \tag{22}$$

$$\bar{\xi}(a,x) \propto \left[\bar{\xi}_L(a,l)\right]^{D/(D-2)} \qquad \text{(quasi-linear)} \tag{23}$$

$$\bar{\xi}(a,x) \propto \left[\bar{\xi}_L(a,l)\right]^{D/2(D-2)} \qquad \text{(non-linear)} \tag{24}$$

In the case of 3 dimensions, the above equations reduce to $\bar{\xi}(a,x) \propto \left[\bar{\xi}_L(a,l)\right]^3$ and $\left[\bar{\xi}_L(a,l)\right]^{3/2}$ in the quasi-linear and non-linear regimes, respectively; these are in reasonable agreement with simulations ([2], [5]; see, however, [6]). Thus, scaling laws appear to be a generic feature of gravitational clustering in an expanding background, in all dimensions (except $D = 2$).

The 2-dimensional case is special and different from all other dimensions, as, in this case, equation (9) gives

$$\frac{d^2R}{dt^2} = 0 \tag{25}$$

i.e. the correlation function does not evolve at all in 2-D and no structures are formed. The special nature of the $D = 2$ case appears to arise from the structure of the Poisson equation, due to the fact that it contains derivatives of the second order. It has been noted earlier that scaling relations *can* arise in 2-D numerical simulations of gravitational clustering ([7]). These simulations, however, do not consider the case of 2-D particles evolving in a 2-D background but instead treat the system as consisting of a set of infinitely long, parallel needles, in a 3-D expanding background, with particles considered as arising at the intersections of the needles with any plane orthogonal to them. The mass elements in the needles interact via the usual $1/r^2$ force; however, the interaction between the needles themselves is governed by a $1/r$ force. The simulations consider the clustering of the needles by taking a slice through a plane orthogonal to them. This situation is, of course, clearly *anisotropic* as clustering is considered as taking place in 2 dimensions while the background Universe expands in 3 dimensions. In this case, one can show numerically ([1]) that $R_M = l/\delta_i$ in the quasi-linear regime and arrive at the relevant scaling relations by an analysis similar to the above.
We finally consider the case of 2-D particles in a 2-D expanding background and *assume* a force law

$$\frac{d^2R}{dt^2} = -\frac{GM}{R} \tag{26}$$

The above form, of course, *cannot* be obtained in a self-consistent manner, by taking limits of the Einstein equations for an expanding Universe; however, it has the form $d^2R/dt^2 \propto R^{-1}$, the correct dependence for $D = 2$ in equation (9). The energy integral of equation (26) is

$$E = \frac{1}{2}\left(\frac{dR}{dt}\right)^2 - GM \ln R \tag{27}$$

The mass M, enclosed within the shell, is given by

$$M = (1+\delta_i)\rho_b \pi l^2 \tag{28}$$

where l is the initial shell radius. Here, we will assume $\rho_b \propto t^{-2}$, the usual result for D dimensions. Setting $dR/dt = H_i l$ initially, and using $H_i \propto 1/t_i$, we obtain

$$R_M = l \exp\left[-A(1+\delta_i)\right] \tag{29}$$

where A is a constant; *i.e.* $R_M/l = const.$, to leading order in δ_i. Again taking the final effective radius, of the 2-dimensional shell, as proportional to R_M, the final mean correlation function is given by

$$\bar{\xi}(x) \propto \rho \propto \frac{M}{x^2} \propto \frac{l^2}{l^2} \propto \text{const.} \tag{30}$$

Thus, the correlation function does not grow (to first order) until much after turnaround. This is an interesting result which needs to be verified by simulations.
The above situation is similar to the case of a 3-D network of parallel cosmic strings (space around them is flat and $\rho_b \propto a^{-2}$). Structure will clearly *not* grow if the network is expanding uniformly; however, if the strings had some initial peculiar velocities, then two strings which pass on either side of a third would move towards each other because the angle around the third string is less than 2π.

III. Conclusions

In the present paper, the standard paradigms of scale-invariant radial collapse and stable clustering have been used to derive scaling relations connecting the exact mean two-point correlation function and the linear mean correlation function, for a D-dimensional expanding Universe. The existence of scaling laws is found to be a generic feature of gravitational clustering in an expanding background, in all dimensions, except $D = 2$. Further, the scaling laws derived here are in agreement, in the 3-dimensional case, with scaling relations obtained earlier from N-body simulations ($\bar{\xi} \propto \bar{\xi}_L^3$ and $\bar{\xi} \propto \bar{\xi}_L^{3/2}$, in the quasi-linear and non-linear regimes, respectively.). Finally, we have considered the case of clustering of 2-D particles in a 2-dimensional expanding background, governed by a force -GM/R, and show, by a similar analysis, that the correlation function does not grow, to first order, until much after turnaround of any shell.

---

* paddy@iucaa.ernet.in
† nissim@ncra.tifr.res.in

[1] J. A. Fillmore and P. Goldreich, ApJ, **281**, 1 (1984)
[2] A. J. S. Hamilton, P. Kumar, E. Lu and A. Matthews, ApJ, **374**, L1 (1991)
[3] E. Bertschinger, ApJS, **58**, 1 (1985)
[4] T. Padmanabhan, MNRAS, **278**, L29 (1996)
[5] T. Padmanabhan, R. Cen, J. P. Ostriker and F. J. Summers., ApJ, **466**, 604 (1996)
[6] J. A. Peacock and S. J. Dodds, MNRAS, 280, L19 (1996)
[7] J. S. Bagla, S. Engineer and T. Padmanabhan, ApJ, **495**, 28 (1998)
[8] S. Engineer, K. Srinivasan and T. Padmanabhan, ApJ, **512**, 1 (1999)
[9] P. J. E. Peebles, The Large Scale Structure of the Universe, Princeton NJ (1980)